Notes on Combinatorial Identities and Pascal's Triangle
ESE (English for Science and Engineering)
Autumn Semester 2008-2009, Sophia University
RMMarcelo
January 6, 2009

COMBINATORIAL IDENTITIES AND PASCAL'S TRIANGLE

In this chapter, we look at some combinatorial identities that arise naturally from the Pascal's Triangle.

1. Definition of the Pascal's Triangle

Recall that for any $n, k \in \mathbb{Z}$, with $0 \le k \le n$, we define

$$\binom{n}{k} = C(n, k) = \text{ the number of } k\text{-subsets of an } n\text{-set.}$$

From the previous chapter, we have shown the following identities:

Identity 1: $\binom{n}{0} = \binom{n}{n} = 1$

Identity 2: $\binom{n}{1} = n$

Identity 3: $\binom{n}{k} = \binom{n}{n-k}$ for each k, $0 \le k \le n$

We define the Pascal's triangle to be the (infinite) triangle formed as follows:

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \; \binom{1}{1} \\
\binom{2}{0} \; \binom{2}{1} \; \binom{2}{2} \\
\binom{3}{0} \; \binom{3}{1} \; \binom{3}{2} \; \binom{3}{3} \\
\binom{4}{0} \; \binom{4}{1} \; \binom{4}{2} \; \binom{4}{3} \; \binom{4}{4} \\
\binom{5}{0} \; \binom{5}{1} \; \binom{5}{2} \; \binom{5}{3} \; \binom{5}{4} \; \binom{5}{5} \\
\binom{6}{0} \; \binom{6}{1} \; \binom{6}{2} \; \binom{6}{3} \; \binom{6}{4} \; \binom{6}{5} \; \binom{6}{6} \\
\binom{7}{0} \; \binom{7}{1} \; \binom{7}{2} \; \binom{7}{3} \; \binom{7}{4} \; \binom{7}{5} \; \binom{7}{6} \; \binom{7}{7} \\
\binom{8}{0} \; \binom{8}{1} \; \binom{8}{2} \; \binom{8}{3} \; \binom{8}{4} \; \binom{8}{5} \; \binom{8}{6} \; \binom{8}{7} \; \binom{8}{8} \\
\binom{9}{0} \; \binom{9}{1} \; \binom{9}{2} \; \binom{9}{3} \; \binom{9}{4} \; \binom{9}{5} \; \binom{9}{6} \; \binom{9}{7} \; \binom{9}{8} \; \binom{9}{9}
\end{array}$$

Using the identities above and the formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

we could then obtain the following for the top 10 rows of the Pascal's triangle.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1
\end{array}$$

We will label the rows of the Pascal's triangle starting with Row 0, then Row 1, etc. Thus, Row n consists of the integers

$$\binom{n}{0} \; \binom{n}{1} \; \binom{n}{2} \; \cdots \binom{n}{n-2} \; \binom{n}{n-1} \; \binom{n}{n}.$$

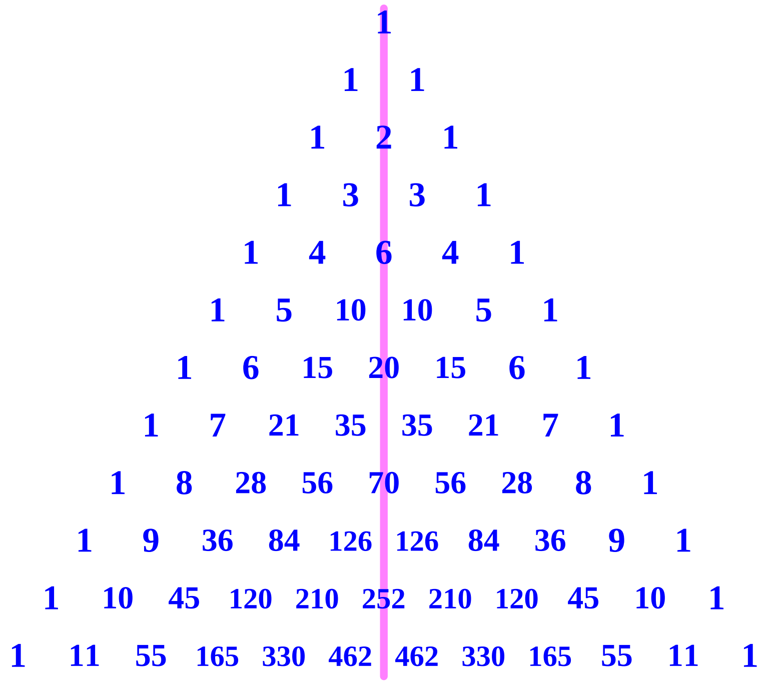
Row 0 : **1**
Row 1 : **1 1**
Row 2 : **1 2 1**
Row 3 : **1 3 3 1**
Row 4 : **1 4 6 4 1**
Row 5 : **1 5 10 10 5 1**
Row 6 : **1 6 15 20 15 6 1**
Row 7 : **1 7 21 35 35 21 7 1**

The three identities above can be translated to the following properties of Pascal's triangle:

Property 1: Each row starts and ends with a 1.

Property 2: The second number in Row n is n.

Property 3: The entries in each row of the Pascal's triangle are symmetric with respect to the center.



Using more identities, we shall prove additional properties of the Pascal's triangle. We first enumerate the properties, and then we translate them to identities.

Property 4: Each number (aside from the first and the last in the row) is obtained by adding the two numbers immediately above it.

Property 5: The numbers in Row n are the coefficients in the expansion of $(x+y)^n$.

Row 0 : 1
Row 1 : 1 1
Row 2 : 1 2 1
Row 3 : 1 3 3 1
Row 4 : 1 4 6 4 1
Row 5 : 1 5 10 10 5 1
Row 6 : 1 6 15 20 15 6 1
Row 7 : 1 7 21 35 35 21 7 1

Examples:

$$\begin{aligned}
(x+y)^1 &= 1\cdot x + 1\cdot y \\
(x+y)^2 &= 1\cdot x^2 + 2\cdot xy + 1\cdot y^2 \\
(x+y)^3 &= 1\cdot x^3 + 3\cdot x^2y + 3\cdot xy^2 + 1\cdot y^3 \\
(x+y)^4 &= 1\cdot x^4 + 4\cdot x^3y + 6\cdot x^2y^2 + 4\cdot xy^3 + 1\cdot y^4
\end{aligned}$$

Property 6: The sum of the numbers in Row n is 2^n.

Row 0 : $1 = 1 = 2^0$

Row 1 : $1 + 1 = 2 = 2^1$

Row 2 : $1 + 2 + 1 = 4 = 2^2$

Row 3 : $1 + 3 + 3 + 1 = 8 = 2^3$

Row 4 : $1 + 4 + 6 + 4 + 1 = 16 = 2^4$

Row 5 : $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$

Row 6 : $1 + 6 + 15 + 20 + 15 + 6 + 1 = 64 = 2^6$

Row 7 : $1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128 = 2^7$

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

Property 7: The result of alternately adding and subtracting numbers in each row (except Row 0) is 0.

1

$1 - 1 = 0$

$1 - 2 + 1 = 0$

$1 - 3 + 3 - 1 = 0$

$1 - 4 + 6 - 4 + 1 = 0$

$1 - 5 + 10 - 10 + 5 - 1 = 0$

$1 - 6 + 15 - 20 + 15 - 6 + 1 = 0$

$1 - 7 + 21 - 35 + 35 - 21 + 7 - 1 = 0$

$1 - 8 + 28 - 56 + 70 - 56 + 28 - 8 + 1 = 0$

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

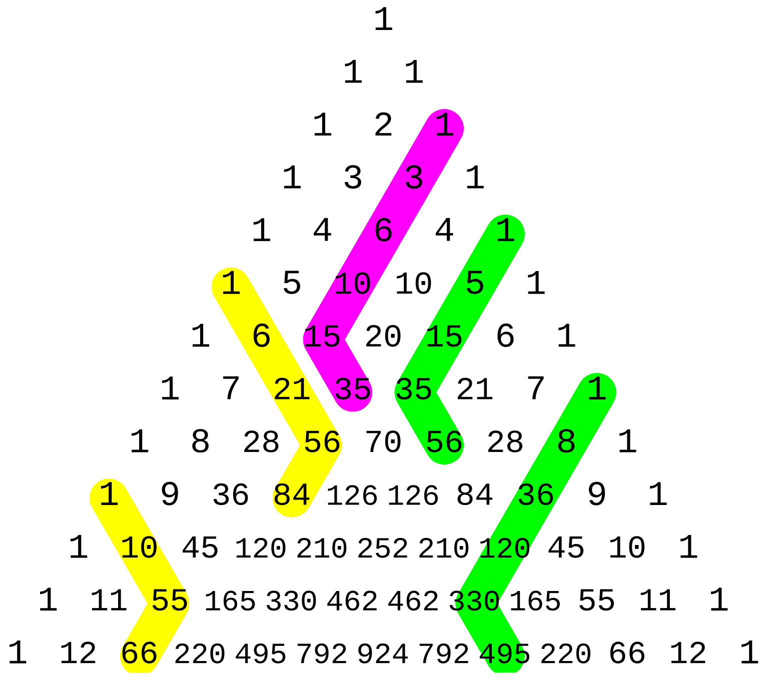
1 11 55 165 330 462 462 330 165 55 11 1

Property 8: If the numbers in Row n are multiplied by decreasing powers of 10 and then the products are added, we obtain 11^n.

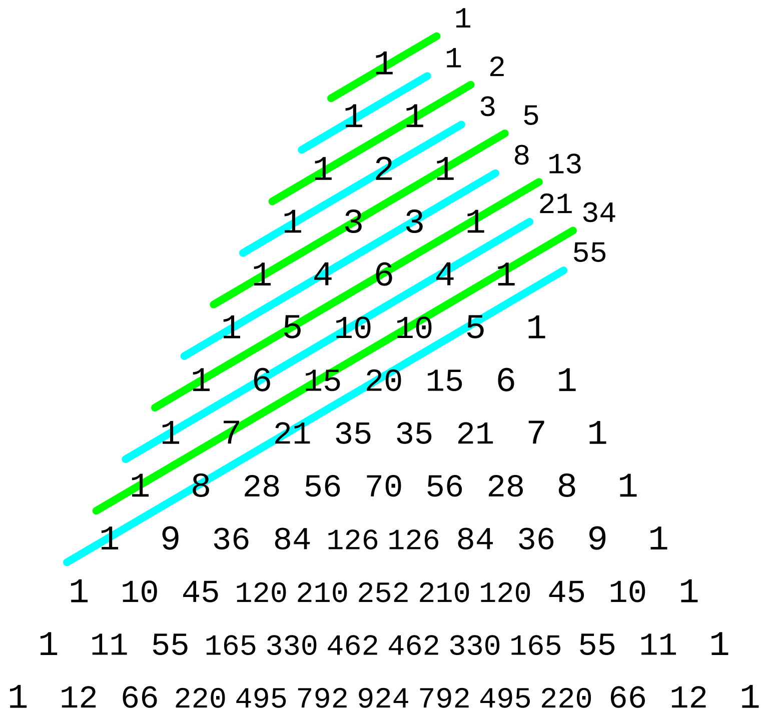
Examples:

$$\begin{aligned}
1 \cdot 10^1 + 1 \cdot 10^0 &= 10 + 1 &&= 11^1 \\
1 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0 &= 100 + 20 + 1 &&= 11^2 \\
1 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10^1 + 1 \cdot 10^0 &= 1000 + 300 + 30 + 1 &&= 11^3 \\
1 \cdot 10^4 + 4 \cdot 10^3 + 6 \cdot 10^2 + 4 \cdot 10^1 + 1 \cdot 10^0 &= 10000 + 4000 + 600 + 40 + 1 &&= 11^4
\end{aligned}$$

Property 9: If “hockey sticks” are formed as below, then the number on the lowest row equals the sum of the other numbers on the stick.



Property 10: The Fibonacci sequence can be obtained by forming sums of the numbers in the diagonals shown below:



(*Note:* The Fibonacci sequence is the sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 99, \ldots$ It begins with two 1’s and the rest are obtained by forming the sum of the two terms before it.)

Property 11: In any hexagon formed as shown below, the product of the numbers in one triangle is equal to the product of the numbers in the other triangle.

$$\begin{array}{c}
1\\
1 \quad 1\\
1 \quad 2 \quad 1\\
1 \quad 3 \quad 3 \quad 1\\
1 \quad 4 \quad 6 \quad 4 \quad 1\\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1\\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1\\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1\\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1\\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1
\end{array}$$

Property 12: The sum of the squares of the entries in Row n equals the entry in the center of Row $2n$.

Examples:

$$\begin{aligned}
1^2 &= 1\\
1^2+1^2 &= 2\\
1^2+2^2+1^2 &= 6\\
1^2+3^2+3^2+1^2 &= 20\\
1^2+4^2+6^2+4^2+1^2 &= 70
\end{aligned}$$

2. Identities 4 to 12

Property #4 above, says that we can obtain the value of each entry (that is, $\binom{n}{k}$) by adding the two entries directly above it (that is, $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$). We can then translate this property into the following identity:

Identity 4: (Pascal's Identity) For any $n, k \in \mathbb{Z}$, with $1 \le k \le n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof: Let S be a set with n elements. Then, $\displaystyle\binom{n}{k}$ is the number of k-subsets of S. Now, suppose a is any element of S. Then the k-subsets of S can be grouped into two classes: those that contain a and those that don't. The number of k-subsets that contain a is clearly equal to $\displaystyle\binom{n-1}{k-1}$ (since each k-subset containing a corresponds to a $(k-1)$-subset of $S \setminus \{a\}$ and $|S \setminus \{a\}| = n-1$.) while the number of k-subsets that don't contain a is equal to $\displaystyle\binom{n-1}{k}$.

Property #5 follows from the Binomial Theorem:

Identity 5: (Binomial Theorem) For any $n \in \mathbb{Z}$, $n \ge 1$,

$$\begin{aligned}
(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-r}x^r y^{n-r} + \cdots \\
+ \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.
\end{aligned}$$

Proof: By definition, $(x+y)^n = (x+y)(x+y)\cdots(x+y)$ (n factors). Each $(x+y)$ contributes either $1x^1$ or $1y^1$ in the product, so every term in the expansion of $(x+y)^n$ must have the form $x^r y^{n-r}$ for some $r \in \mathbb{Z}$, $0 \le r \le n$. Now the number of times that $x^r y^{n-r}$ is formed is exactly the number of ways to select x exactly r times. Since there are n such $(x+y)$'s to choose from, this number is exactly $\binom{n}{r}$ or $\binom{n}{n-r}$, completing the proof.

As a consequence of the Binomial theorem, we obtain the identities corresponding to Properties 6, 7 and 8. To prove Identity 6, we make the substitution $x = y = 1$. For Identity 7, we let $x = 1$ and $y = -1$. Identity 8 follows if we let $x = 10$ and $y = 1$.

Identity 6: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n} = 2^n.$

Identity 7: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$

Identity 8: $\binom{n}{0}10^n + \binom{n}{1}10^{n-1} + \cdots + \binom{n}{n-1}10^1 + \binom{n}{n}10^0 = 11^n.$

Property 9 follows from the following identities called Chuh Shih Chieh's identities.

Identity 9:

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{k+\ell}{k} = \binom{k+\ell+1}{k+1}, \text{ and}$$

$$\binom{k}{0} + \binom{k+1}{1} + \binom{k+2}{2} + \cdots + \binom{k+\ell}{\ell} = \binom{k+\ell+1}{\ell}.$$

Example: $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3}$

$$\begin{aligned}\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} &= \binom{3}{3} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} \\ &= \binom{4}{3} + \binom{4}{2} + \binom{5}{2} \\ &= \binom{5}{3} + \binom{5}{2} \\ &= \binom{6}{3}.\end{aligned}$$

Property 12 follows from the following identity:

Identity 12: $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}.$

The last identity is a corollary of the following more general identity, if we make the substitution $m = n$ and $k = n$, and use Identity 3.

Theorem: (Vandermonde Identity) For any integers $n, m, k \ge 0$, with $0 \le k \le n, m$,

$$\binom{n}{0}\binom{m}{k-0} + \binom{n}{1}\binom{m}{k-1} + \binom{n}{2}\binom{m}{k-2} + \cdots + \binom{n}{k}\binom{m}{k-k} = \binom{n+m}{k}. \qquad \text{(V)}$$

Proof: Let $S = \{1, 2, \ldots, n\}$, $T = \{n+1, n+2, \ldots, n+m\}$, and $U = S \cup T$. Then the right hand side of (V) is the number of k-subsets of U. On the other hand, the left side counts the same thing. Note that the left hand side counts the number of k-subsets of U where i of the elements come from S and $k-i$ come from T, for $i = 0, 1, \ldots, k$.